8/3/25

Problem 1

$$\frac{dy}{dx} + y = 2 \quad , \quad y(0) = 0$$

$$\frac{dy}{dx} = 2 - y$$

$$-\ln|2-y| = x + C_1$$

$$|2-y| = Ce^{-x}$$

$$\therefore y = 2 - Ce^{-x}$$

$$y(0) = 0 \Rightarrow \therefore y = 2(1 - e^{-x})$$

Problem 2

$$xy' - y = x \quad , \quad y(1) = 7$$

$$x\frac{dy}{dx} - y = x \qquad \frac{d}{dx}(uy) = u'y + uy'$$

$$\frac{dy}{dx} - \frac{y}{x} = 1$$

$$uy' - u\cdot\frac{1}{x}y = u$$

$$uy' - u\cdot\frac{1}{x}y = u'y + uy'$$

$$-\frac{u}{x} = u'$$

$$\int \frac{1}{u}\,du = -\int \frac{1}{x}\,dx$$

$$\ln|u| = -\ln|x| + C_1$$

$$u = Ce^{-\ln|x|}$$

$$\therefore u = e^{-\ln|x|} = \frac{1}{x}$$

$$\frac{d}{dx}(uy) = u$$

$$\int \frac{d}{dx}(uy)\,dx = \int \frac{1}{x}\,dx$$

$$u(x)y(x) = \ln|x| + C$$

$$\therefore\ y = x\ln|x| + Cx$$

$$y(1) = 7 \Rightarrow\quad 7 = C$$

$$\therefore\ y = x(\ln|x| + 7)$$

Problem 3

$$y' = 1 + x + y + xy, \quad y(0) = 0$$

$$y' - (1+x)y = 1 + x$$

$$uy' - u(1+x)y = u + ux$$

$$uy' + u'y = uy' - u(1+x)y$$

$$u'y = -u(1+x)y$$

$$u' = -u(1+x)$$

$$\int -\frac{1}{u}\, du = \int (1+x)\, dx$$

$$-\ln|u| = x + \frac{x^2}{2} + C_1$$

$$|u| = e^{-x-\frac{x^2}{2}} \cdot e^{-C_1}$$

$$u = Ce^{-x-\frac{x^2}{2}}$$

$$\therefore u = e^{-x-\frac{x^2}{2}}$$

$$\int \frac{d}{dx}(uy)\,dx = \int 1 + x \; dx$$

$$uy = x + \frac{x^2}{2} + C$$

$$y = e^{x\left(1+\frac{x}{2}\right)}\left(x + \frac{x^2}{2} + C\right)$$

$$y(0) = 0 \Rightarrow 0 = C$$

$$\therefore \; y = e^{x\left(1+\frac{x}{2}\right)}\left(x + \frac{x^2}{2}\right)$$

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Problem 4

$$10\ g/\ell \times 10\ \ell/min = 100\ g/min$$

a)

$$\frac{dc}{dt} = \frac{100\ g/min}{100{,}000\ \ell} - \frac{10\ c(t)\ g/min}{100{,}000\ \ell}$$

$$= \frac{1}{1000} - \frac{1}{10{,}000}\, c(t)$$

$$\frac{dc(t)}{dt} + \frac{1}{10000}\, c(t) = \frac{1}{1000}$$

b) $$u\dot{c} + \frac{u}{10000}\, c = \frac{u}{1000}$$

$$\dot{u}c + u\dot{c} = u\dot{c} + \frac{u}{10000}\, c$$

$$\dot{u}\, c = \frac{u}{10000}\, c$$

$$\dot{u} = \frac{u}{10000}$$

$$\int \frac{1}{u}\frac{du}{dt}\,dt = \int \frac{1}{10000}\,dt$$

$$\ln|u| = \frac{t}{10000} + C.$$

$$\therefore u = e^{\frac{t}{10000}}$$

$$\frac{d}{dt}(uc) = \frac{u}{1000}$$

$$uc = 10000\,\frac{e^{\frac{t}{10000}}}{1000} + C$$

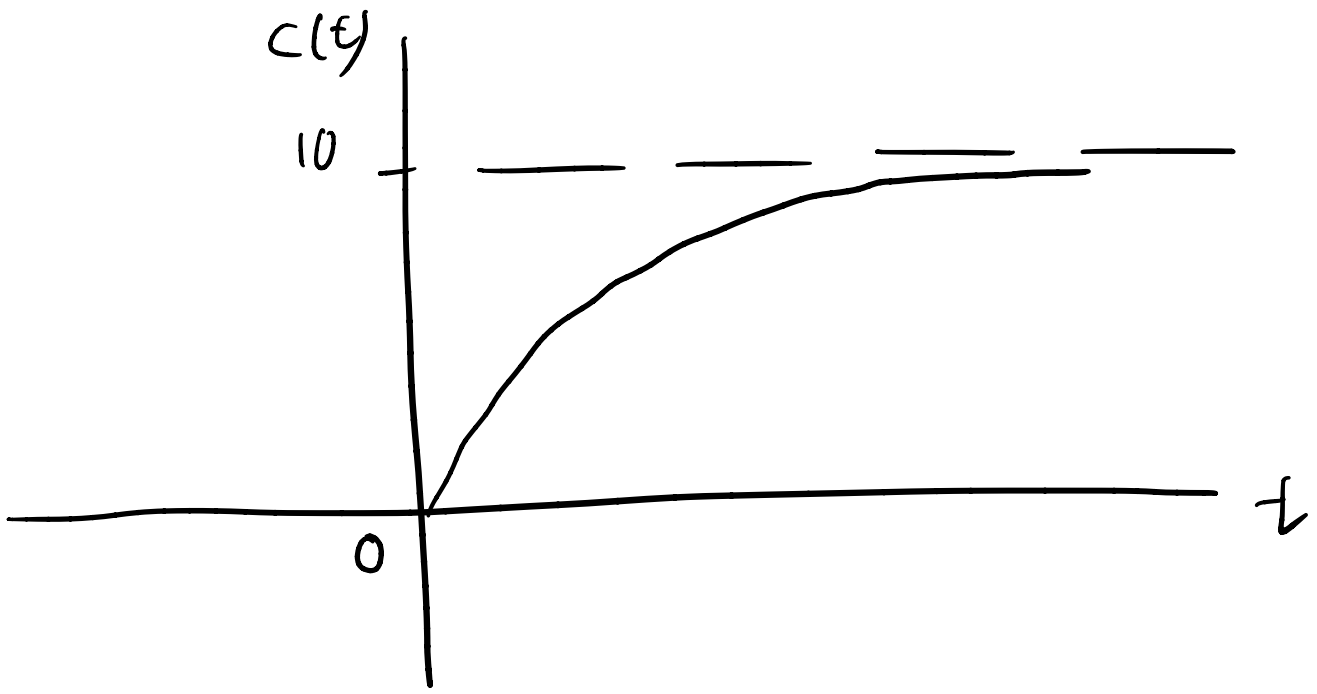
$$c = \frac{10\,e^{\frac{t}{10000}} + C}{e^{\frac{t}{10000}}}$$

$$c(t) = 10 + Ce^{-\frac{t}{10000}}$$

$$c(0) = 0 \Rightarrow \quad 0 = 10 + C$$

$$C = -10$$

$$\therefore c(t) = 10\left(1 - e^{-\frac{t}{10000}}\right)$$



c) $5 = 10\left(1 - e^{-\frac{t}{10000}}\right)$

$$\frac{1}{2} - 1 = -e^{-\frac{t}{10000}}$$

$$e^{-\frac{t}{10000}} = \frac{1}{2}$$

$$-\frac{t}{10000} = \ln 1 - \ln 2$$

$$t = 10000 \ln 2$$

d) Concentration will get infinitely close to but never reach 10 g / litre.